Construction of self-adaptive moving mesh methods by hodograph transformation

Baofeng Feng

School of Mathematical and Statistical Sciences,
The University of Texas Rio Grande Valley

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Outline

- A class of nonlinear wave equations with loop, cuspon and breather solutions
- Integrable discretization and integrable self-adaptive moving mesh methods
- Construction of self-adaptive moving mesh methods by hodograph transformation
- Summary and further topics

Collaborators:

K. Maruno (Waseda University), Y. Ohta (Kobe University),

T. Matsuo (The Univ. of Tokyo), W.-H. Sheu (National Taiwan University)

Motivation

Structure-preserving and moving mesh numerical methods

- Multi-symplectic integrator

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- Invariant-preserving integrator

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- Self-adaptive moving mesh methods

B. Leimkuhler and S. Reich, *Simulating Hamiltonian Dynamics* (Cambridge University Press, 2004).

D. Furihata and T. Matsuno, *Discrete Variational Derivative Method* (CRC Press, Boca Raton, FL, 2011)

W. Huang and R. D. Russell, *Adaptive Moving Mesh Methods* (Springer, FL, 2011).

Semi-discrete NLS equation (Ablowitz-Ladik lattice) to the Nonlinear Schrödinger (NLS) equation $\mathrm{i}q_t + q_{xx} + 2|q|^2 q = 0$,

$$\mathrm{i}\frac{dq_k}{dt} + \frac{q_{k+1} - 2q_k + q_{k-1}}{h^2} + |q_k|^2(q_{k+1} + q_{k-1}) = 0,$$

where $q_k \approx q(kh, t)$. D.A. Karpeev, C.M. Schober, Math. and Compt. Simul. 56 (2001) 1456

C.M. Schober, Phys. Lett. A 259 (1999) 1401.

Fully discrete sine-Gordon equation $u_{xt} = \sin u$

$$\frac{1}{ab}\sin\frac{u_{k+1}^{l+1} - u_{k+1}^{l} - u_{k}^{l+1} + u_{k}^{l}}{4} = \sin\frac{u_{k+1}^{l+1} + u_{k+1}^{l} + u_{k}^{l+1} + u_{k}^{l}}{4}.$$

where $u_k^l \approx u(ka, lb)$.

M. J. Ablowitz, B. M. Herbst, and C. M. Schober, Phys. Rev. Lett. 71, 2683 (1993).

M. J. Ablowitz, B. M. Herbst, and C. M. Schober, J. Comput. Phys. 126, 299 (1996), 131, 354 (1997).

The Camassa-Holm equation

$$u_t + 2\kappa^2 u_x - u_{txx} + 3uu_x = 2u_x u_{xx} + uu_{xxx}$$

$$m_t + 2mu_x + um_x = 0\,, \quad m = \kappa^2 + u - u_{xx}$$

R. Camassa, D.D. Holm, Phys. Rev. Lett. 71 (1993) 1661

Inverse scattering transform, A. Constantin, (2001)

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R. Camassa, D.D. Holm, Phys. Rev. Lett. 71 (1993) 1661

Inverse scattering transform, A. Constantin, (2001)

Short wave limit: $t \to \epsilon t, x \to x/\epsilon, u \to \epsilon^2 u$

The Hunter-Saxton equation

$$u_{txx} - 2\kappa^2 u_x + 2u_x u_{xx} + uu_{xxx} = 0$$

Hunter, & Saxton (1991): Nonlinear orientation waves in liquid crystals

Hunter & Zheng (1994): Lax pair, bi-Hamiltonian structure

FMO (2010): Integrable semi- and fully discretizations

The Degasperis-Procesi equation

$$u_t + 3\kappa^3 u_x - u_{txx} + 4uu_x = 3u_x u_{xx} + uu_{xxx}\,,$$

$$m_t + 3mu_x + um_x = 0\,, \quad m = \kappa^3 + u - u_{xx}$$

A. Degasperis, M. Procesi, (1999)

Degasperis, Holm, Hone (2002)

N-soliton solution, Matsuno (2005)

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Degasperis, Holm, Hone (2002)

N-soliton solution, Matsuno (2005)

Short wave limit:

$$u_{txx} - 3\kappa^3 u_x + 3u_x u_{xx} + uu_{xxx} = 0$$

$$u_{tx} - 3\kappa^3 u + \frac{1}{2}(u^2)_{xx} = 0$$

- **Reduced Ostrovsky equation**, L.A. Ostrovsky, *Okeanologia* 18, 181 (1978).
- **Vakhnenko equation**, V. Vakhnenko, *JMP*, 40, 2011 (1999)

The short pulse equation

$$u_{xt} = u + \frac{1}{6}(u^3)_{xx}$$

- Schäfer & Wayne(2004): Derived from Maxwell equation on the setting of ultra-short optical pulse in silica optical fibers.
- Sakovich & Sakovich (2005): A Lax pair of WKI type, linked to sine-Gordon equation through hodograph transformation;
- Brunelli (2006) Bi-Hamiltonian structure, Phys. Lett. A 353, 475478
- Matsuno (2007): Multisoliton solutions through Hirota's bilinear method
- FMO (2010): Integrable semi- and fully discretizations.

Complex short pulse equation

$$q_{xt} + q + \frac{1}{2}(|q|^2 q_x)_x = 0$$

- The complex short pulse equation, which can be derived from Maxwell equation, is more natural and appropriate than short pulse equation in describing the propagation of the ultra-short pulses. It is an analogue of the NLS equation for the ultra-short pulses.
- It is integrable and and has exact N-envelop soliton solution.
- B.-F (2015) *Physica D* 297, 62-75

Coupled complex short pulse equation

$$q_{1,xt} + q_1 + \frac{1}{2}\left((|q_1|^2 + B|q_2|^2)q_{1,x}\right)_x = 0\,,$$

$$q_{2,xt} + q_2 + \frac{1}{2}\left((|q_2|^2 + B|q_1|^2)q_{2,x}\right)_x = 0\,.$$

- The parameter B is related to the ellipticity angle θ as

$$B = \frac{2 + 2\sin^2\theta}{2 + \cos^2\theta}\,.$$

- For a linearly birefringent fiber ($\theta = 0$), $B = \frac{2}{3}$, for a circularly birefringent fiber ($\theta = \pi/2$), $B = 2$. Only when $B = 1$, it is integrable.

The generalized sine-Gordon equation

$$u_{xt} = (1 + \sigma\partial_{xx}) \sin u \,, \quad \sigma = \pm 1$$

- Proposed by A. Fokas through a bi-Hamiltonian method (1995)
- Matsuno gave a variety of soliton solutions such as kink, loop and breather solutions (2011)
- For $\sigma = 1$, under the short wave limit $\bar{u} = u/\epsilon, \bar{x} = (x - t)/\epsilon$, $\bar{t} = \epsilon t$, it converges to the short pulse equation.
- Under the long wave limit $\bar{u} = u, \bar{x} = \epsilon x$, $\bar{t} = t/\epsilon$, it converges to the sine-Gordon equation.

The link of the short pulse equation to the coupled dispersioless equation

$$u_{xt} = u + \frac{1}{2}(u^2 u_x)_x \,, \quad \partial_x(\partial_t - \frac{1}{2}u^2\partial_x)u = u \,.$$

It can be easily shown that

$$\left(\sqrt{1+u_x^2}\right)_t - \left(\frac{1}{2}u^2\sqrt{1+u_x^2}\right)_x = 0$$

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It can be easily shown that

$$\left(\sqrt{1+u_x^2}\right)_t - \left(\frac{1}{2}u^2\sqrt{1+u_x^2}\right)_x = 0$$

Letting $\rho = \left(1+u_x^2\right)^{1/2}$, we can define a hodograph transformation

$$dy = \rho dx - \frac{1}{2}u^2 \rho dt \,, \quad ds = dt \,,$$

or

$$\partial_x = \rho^{-1}\partial_y, \quad \partial_t = \partial_s + \frac{1}{2}u^2\rho^{-1}\partial_y$$

which leads to

$$\begin{cases} u_{ys} = \rho u, \\ \rho_s + \frac{1}{2}(u^2)_y = 0 \end{cases}$$

The Lax pair for the CD equation

$$\Psi_y = U\Psi, \quad \Psi_s = V\Psi\,,$$

$$U = -\mathrm{i}\lambda \begin{pmatrix} \rho & u_y \\ u_y & -\rho \end{pmatrix}, \quad V = \begin{pmatrix} \frac{\mathrm{i}}{4\lambda} & -\frac{u}{2} \\ \frac{u}{2} & -\frac{\mathrm{i}}{4\lambda} \end{pmatrix}$$

The compatibility condition $U_t - V_x - UV + VU = 0$ gives the CD equation. Then we get the Lax pair for the SP equation through the hodograph transformation $\partial_y = \rho\partial_x\,,\ \partial_s = \partial_t - 1/2u^2\partial_x$

$$\Psi_x = U\Psi, \quad \Psi_t = V\Psi\,,$$

$$U = -\mathrm{i}\lambda \begin{pmatrix} 1 & u_x \\ u_x & -1 \end{pmatrix}$$

$$V = \begin{pmatrix} \frac{\mathrm{i}}{4\lambda} - \frac{\mathrm{i}\lambda}{2}u^2 & -\frac{\mathrm{i}\lambda}{2}u^2u_x - \frac{u}{2} \\ -\frac{\mathrm{i}\lambda}{2}u^2u_x + \frac{u}{2} & -\frac{\mathrm{i}}{4\lambda} + \frac{\mathrm{i}\lambda}{2}u^2 \end{pmatrix}$$

Theorem

The bilinear equations

$$D_sD_yf\cdot g=fg\,,\quad D_s^2f\cdot f=\frac{1}{2}g^2\,,$$

where

$$D_s^nD_y^mf\cdot g=\left(\frac{\partial}{\partial s}-\frac{\partial}{\partial s'}\right)^n\left(\frac{\partial}{\partial y}-\frac{\partial}{\partial y'}\right)^m f(y,s)g(y',s')|_{y=y',s=s'}$$

give the short pulse equation

$$u_{xt}=u+\frac{1}{2}\left(u^2u_x\right)_x$$

through the hodograph and dependent variable transformations

$$x=y-2(\ln f)_s\,,\quad t=s\,,\quad u=\frac{g}{f}$$

Proof of the Theorem (I)

$$D_y f \cdot g = f_y g - f g_y \,, \quad D_s D_y f \cdot g = f_{sy} g - f_s g_y - f_y g_s + f g_{sy} \,,$$
$$\frac{D_s D_y f \cdot g}{f^2} = \left(\frac{g}{f}\right)_{sy} + 2(\ln f)_{ys} \frac{g}{f}$$
$$D_s^2 f \cdot f = 2 f_{sy} f - 2 f_s f_y \,, \quad \frac{D_s^2 f \cdot f}{f^2} = 2(\ln f)_{ss}$$

$$\begin{cases} \left(\dfrac{g}{f}\right)_{sy} = (1 - 2(\ln f)_{ys}) \dfrac{g}{f}, \\ 2(\ln f)_{ss} = \dfrac{1}{2} \dfrac{g^2}{f^2} \end{cases}$$

Proof of the Theorem (I)

$$D_y f\cdot g = f_y g - f g_y\,, \quad D_sD_y f\cdot g = f_{sy}g - f_s g_y - f_y g_s + f g_{sy}\,,$$

$$\frac{D_sD_y f\cdot g}{f^2} = \left(\frac{g}{f}\right)_{sy} + 2(\ln f)_{ys}\frac{g}{f}$$

$$D_s^2 f\cdot f = 2f_{sy}f - 2f_s f_y\,, \quad \frac{D_s^2 f\cdot f}{f^2} = 2(\ln f)_{ss}$$

$$\begin{cases} \left(\frac{g}{f}\right)_{sy} = (1-2(\ln f)_{ys})\frac{g}{f}, \\ 2(\ln f)_{ss} = \frac{1}{2}\frac{g^2}{f^2} \end{cases}$$

Let $u = g/f$, $\rho = 1-2(\ln f)_{ys}$, we have

$$\begin{cases} u_{ys} = \rho u, \\ \rho_s + \frac{1}{2}(u^2)_y = 0 \end{cases}$$

Proof of the Theorem (II)

Recall the hodograph transformation

$$x = y - 2(\ln f)_s\,, \quad t = s\,,$$

$$\frac{\partial x}{\partial y} = 1 - 2(\ln f)_{ys} = \rho \quad \frac{\partial x}{\partial s} = -2(\ln f)_{ss} = -\frac{1}{2}u^2$$

or

$$\partial_y = \rho\partial_x\,, \quad \partial_s = \partial_t - \frac{1}{2}u^2\partial_x$$

So the coupled dispersionless equation becomes

$$\partial_x(\partial_t - \frac{1}{2}u^2\partial_x)u = u$$

which is exactly the short pulse equation.

The short pulse equation admits multi-soliton solution

$$f = \begin{vmatrix} A & I \\ -I & B \end{vmatrix}_{2N\times 2N}, \quad g = \begin{vmatrix} A & I & \Phi^T \\ -I & B & 0^T \\ 0 & C & 0 \end{vmatrix}_{(2N+1)\times(2N+1)},$$

where the elements defined respectively by

$$a_{ij} = \frac{1}{2(p_i^{-1} + p_j^{-1})} e^{\xi_i + \xi_j}, \quad b_{ij} = \frac{\alpha_i \alpha_j}{2(p_j^{-1} + p_i^{-1})}$$

$$\Phi = (e^{\xi_1}, e^{\xi_2}, \cdots, e^{\xi_N}), \quad C = -(\alpha_1, \alpha_2, \cdots, \alpha_N),$$

with $\xi_i = p_i y + \frac{1}{p_i} s + \xi_{i0}$

Integrable semi-discrete short pulse equation

Theorem

By discrete hodograph transformation $x_k = 2ka - 2(\ln f_k)_s, \quad t = s$ *and dependent variable transformations* $u_k = \frac{g_k}{f_k}$*, the bilinear equations*

$$\begin{cases} \frac{1}{a} D_s(g_{k+1} \cdot f_k - g_k \cdot f_{k+1}) = g_{k+1} f_k + g_k f_{k+1}\,, \\ D_s^2 f_k \cdot f_k = \frac{1}{2} g_k^2\,. \end{cases}$$

give an integrable semi-discrete short pulse equation

$$\begin{cases} \frac{d}{ds}(u_{k+1} - u_k) = \frac{1}{2}(x_{k+1} - x_k)(u_{k+1} + u_k)\,, \\ \frac{d}{ds}(x_{k+1} - x_k) = -\frac{1}{2}\left(u_{k+1}^2 - u_k^2\right) \end{cases}$$

Proof of the semi-discrete SP equation (I)

Denote $f_k = f(ka, s)$, $g_k = g(ka, s)$, and consider

$$
\begin{aligned}
& D_s D_y f \cdot g = f_{sy} g - f_s g_y - f_y g_s + f g_{sy} \,, \\
& \to \frac{f_{k+1,s} - f_{k,s}}{a} g_k - f_{k,s} \frac{g_{k+1} - g_k}{a} \\
& \quad - g_{k,s} \frac{f_{k+1} - f_k}{a} + \frac{g_{k+1,s} - g_{k,s}}{a} f_k \\
& = \frac{1}{a} (g_{k+1,s} f_k - f_{k,s} g_{k+1} - g_{k,s} f_{k+1} + g_k f_{k+1,s}) \\
& = \frac{1}{a} D_s (g_{k+1} \cdot f_k - g_k \cdot f_{k+1})
\end{aligned}
$$

we could propose

$$
\left\{
\begin{array}{l}
\frac{1}{a} D_s (g_{k+1} \cdot f_k - g_k \cdot f_{k+1}) = g_{k+1} f_k + g_k f_{k+1} \,, \\
D_s^2 f_k \cdot f_k = \frac{1}{2} g_k g_k \,.
\end{array}
\right.
$$

Proof of the semi-discrete SP equation (II)

$$\begin{cases} \frac{1}{a} D_s(g_{k+1} \cdot f_k - g_k \cdot f_{k+1}) = g_{k+1} f_k + g_k f_{k+1} \,, \\ D_s^2 f_k \cdot f_k = \frac{1}{2} g_k^2 \,, \end{cases}$$

The second bilinear equation can be rewritten as

$$(\ln f_k)_{ss} = \frac{1}{4} \frac{g_k^2}{f_k^2} = \frac{1}{4} q_k^2 \,.$$

From the hodograph transformation, we have

$$x_{k+1} - x_k = 2a - 2 \left(\ln \frac{f_{k+1}}{f_k} \right)_s \,,$$

it immediately follows

$$\frac{d}{ds}(x_{k+1} - x_k) = -\frac{1}{2} \left(q_{k+1}^2 - q_k^2 \right)$$

Proof of the semi-discrete SP equation (III)

Dividing both sides by $f_{k+1}f_k$, the first bilinear equations can be calculated out by

$$\left(\frac{g_{k+1,s}}{f_{k+1}}-\frac{g_{k,s}}{f_k}\right)-\frac{g_{k+1}f_{k,s}-g_kf_{k+1,s}}{f_{k+1}f_k}=a\left(\frac{g_{k+1}}{f_{k+1}}+\frac{g_k}{f_k}\right)\,,$$

which is recast into

$$\left(\frac{g_{k+1}}{f_{k+1}}-\frac{g_k}{f_k}\right)_s=\left(a-\left(\ln\frac{f_{k+1}}{f_k}\right)_s\right)\left(\frac{g_{k+1}}{f_{k+1}}+\frac{g_k}{f_k}\right)\,.$$

With the use of discrete hodograph and dependent variable transformations, we have

$$\frac{d}{ds}(q_{k+1}-q_k)=\frac{1}{2}(x_{k+1}-x_k)(q_{k+1}+q_k)\,. \qquad (1)$$

Integrable self-adaptive moving mesh method for the short pulse equation

We apply the semi-implicit Euler scheme to the semi-discrete short pulse equation

$$\begin{cases} \frac{d}{ds}(u_{k+1}-u_k) = \frac{1}{2}\delta_k(u_{k+1}+u_k)\,, \\ \frac{d}{ds}(x_{k+1}-x_k) = -\frac{1}{2}(u_{k+1}^2-u_k^2)\,, \end{cases}$$

as follows

$$\begin{cases} p_k^{n+1} = p_k^n + \frac{1}{2}\delta_k^n(u_{k+1}^n+u_k^n)\Delta t\,, \\ \delta_k^{n+1} = \delta_k^n - \frac{1}{2}\left((u_{k+1}^{n+1})^2-(u_k^{n+1})^2\right)\Delta t\,, \end{cases}$$

where $p_k^n = u_{k+1}^n - u_k^n$, $\delta_k^n = x_{k+1}^n - x_k^n$.

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as follows

$$\begin{cases} p_k^{n+1} = p_k^n + \frac{1}{2}\delta_k^n(u_{k+1}^n+u_k^n)\Delta t\,, \\ \delta_k^{n+1} = \delta_k^n - \frac{1}{2}\left((u_{k+1}^{n+1})^2-(u_k^{n+1})^2\right)\Delta t\,, \end{cases}$$

where $p_k^n = u_{k+1}^n - u_k^n$, $\delta_k^n = x_{k+1}^n - x_k^n$.

- Although the semi-implicit Euler is a first-order integrator, it is symplectic which is an appropriate for the Hamiltonian system
- The mesh is evolutive and self-adaptive, so we name it **self-adaptive moving mesh method**.

The complex short pulse equation and the complex coupled dispersionless equation

$$q_{xt} + q + \frac{1}{2}\left(|q|^2 q_x\right)_x = 0\,,$$

gives

$$\left(\sqrt{1+|q|_x^2}\right)_t + \left(\frac{1}{2}|q|^2\sqrt{1+|q|_x^2}\right)_x = 0\,.$$

This leads to a hodograph transformation by defining $\rho = \sqrt{1+|q|_x^2}$. As a result, we obtain the so-called complex coupled dispersionless equation

$$\begin{cases} q_{ys} = \rho q, \\ \rho_s + \dfrac{1}{2}(|q|^2)_y = 0 \end{cases}$$

Konno K, Kakuhata H. J Phys Soc Jpn 1995, 64, 2707; 1996;65:713.

Bilinear equations of the complex short pulse equation

Theorem

The complex short pulse equation

$$q_{xt} + q + \frac{1}{2}\left(|q|^2 q_x\right)_x = 0$$

can be derived from bilinear equations

$$D_s D_y f \cdot g = fg\,, \quad D_s^2 f \cdot f = \frac{1}{2}|g|^2\,,$$

through the hodograph and dependent variable transformations

$$x = y - 2(\ln f)_s\,, \quad t = -s\,, \quad q = \frac{g}{f}$$

Integrable semi-discrete complex short pulse equation

Theorem

The semi-discrete analogue of the complex short pulse equation

$$\begin{cases} \frac{d}{ds}(q_{k+1} - q_k) = \frac{1}{2}(x_{k+1} - x_k)(q_{k+1} + q_k)\,, \\ \frac{d}{ds}(x_{k+1} - x_k) = -\frac{1}{2}\left(|q_{k+1}|^2 - |q_k|^2\right) \end{cases}$$

is decomposed into bilinear equations:

$$\begin{cases} \frac{1}{a}D_s(g_{k+1} \cdot f_k - g_k \cdot f_{k+1}) = g_{k+1}f_k + g_k f_{k+1}\,, \\ D_s^2 f_k \cdot f_k = \frac{1}{2} g_k \bar{g}_k\,. \end{cases}$$

through discrete hodograph transformation and dependent variable transformations $x_k = 2ka - 2(\ln f_k)_s,\ \ t = -s\,, \quad q_k = \frac{g_k}{f_k}$.

Integrable semi-discrete coupled complex short pulse equation

Theorem

The semi-discrete coupled complex short pulse equation

$$\left\{\begin{array}{l} \frac{d}{ds}(q_{1,k+1}-q_{1,k}) = \frac{1}{2}(x_{k+1}-x_k)(q_{1,k+1}+q_{1,k})\,, \\ \frac{d}{ds}(q_{2,k+1}-q_{2,k}) = \frac{1}{2}(x_{k+1}-x_k)(q_{2,k+1}+q_{2,k})\,, \\ \frac{d}{ds}(x_{k+1}-x_k) = -\frac{1}{2}\sum_j(|q_{j,k+1}|^2-|q_{j,k}|^2)\,, \end{array}\right.$$

is decomposed into bilinear equations:

$$\left\{\begin{array}{l} \frac{1}{a}D_s(g^{(i)}_{k+1}\cdot f_k - g^{(i)}_k\cdot f_{k+1}) = g^{(i)}_{k+1}f_k + g^{(i)}_k f_{k+1}\,, \quad i=1,2 \\ D_s^2 f_k\cdot f_k = \frac{1}{2}\left(|g^{(1)}_k|^2+|g^{(2)}_k|^2\right)\,. \end{array}\right.$$

through discrete hodograph transformation and dependent variable transformations $x_k = 2ka - 2(\ln f_k)_s$, $t=-s$, $q_{i,k} = \frac{g^{(i)}_k}{g_k}$.

Integrable self-adaptive moving mesh method for the complex short pulse equation

We apply the semi-implicit Euler scheme to the semi-discrete complex short pulse equation

$$\begin{cases} \frac{d}{dt}(q_{k+1} - q_k) = \frac{1}{2}\delta_k(q_{k+1} + q_k)\,, \\ \frac{d}{dt}(x_{k+1} - x_k) = -\frac{1}{2}(|q|^2_{k+1} - |q|^2_k)\,, \end{cases}$$

as follows

$$\begin{cases} p_k^{n+1} = p_k^n + \frac{1}{2}\delta_k^n(q_{k+1}^n + q_k^n)\Delta t\,, \\ \delta_k^{n+1} = \delta_k^n - \frac{1}{2}\left((|q|_{k+1}^{n+1})^2 - (|q|_k^{n+1})^2\right)\Delta t\,, \end{cases}$$

where $p_k^n = q_{k+1}^n - q_k^n$, $\delta_k^n = x_{k+1}^n - x_k^n$.

Bilinearization to the CH equation

Theorem (F-Maruno-Ohta (2008))

The CH equation

$$m_t + 2mu_x + um_x = 0\,, \quad m = \frac{1}{c} + u - u_{xx}$$

is derived from the bilinear equations

$$(D_y D_s + \frac{1}{c} D_x + 2cD_t) g \cdot h = 0 \quad (2)$$

$$(\frac{1}{2c} D_y + 1) g \cdot h = ff \quad (3)$$

$$(\frac{1}{2} D_y D_s - 1) f \cdot f = -gh \quad (4)$$

through a hodograph transformation $x = 2cy + \ln \frac{g}{h}, \quad t = s$ *and dependent variable transformation* $u = \left(\ln \frac{g}{h}\right)_s$

Proof of the Theorem (I)

$$(\ln gh)_{ys} + ((\ln \frac{g}{h})_y + 2c)((\ln \frac{g}{h})_t + \frac{1}{c}) - 2 = 0\,, \tag{5}$$

$$\frac{1}{2c}(\ln \frac{g}{h})_y + 1 = \frac{f\bar{f}}{gh}\,, \tag{6}$$

$$(\ln f)_{ys} - 1 = -\frac{gh}{f\bar{f}} \tag{7}$$

Proof of the Theorem (I)

$$(\ln gh)_{ys} + ((\ln \frac{g}{h})_y + 2c)((\ln \frac{g}{h})_t + \frac{1}{c}) - 2 = 0\,, \quad (5)$$

$$\frac{1}{2c}(\ln \frac{g}{h})_y + 1 = \frac{ff}{gh}\,, \quad (6)$$

$$(\ln f)_{ys} - 1 = -\frac{gh}{ff} \quad (7)$$

$$(\ln \frac{gh}{ff})_{ys} + ((\ln \frac{g}{h})_y + 2c)((\ln \frac{g}{h})_t + \frac{1}{c}) = 2\frac{gh}{ff} \quad (8)$$

Let $\rho = gh/f^2$, differentiating Eq. (6) with respect to s

$$\frac{1}{2c}u_y = -\frac{\rho_s}{\rho^2}\,, \quad (9)$$

Combining (8) with (6)

$$(\ln \rho)_{ys} + \frac{2c}{\rho}(u + \frac{1}{c}) = 2\rho \quad (10)$$

Proof of the Theorem (II)

From the hodograph transformation

$$\begin{cases} \partial_y = \frac{2c}{\rho}\partial_x \,, \\ \partial_s = \partial_t + u\partial_x \,. \end{cases}$$

Substituting into

$$\begin{cases} (\ln\rho)_s = -u_x, \\ -u_{xx} + u + \frac{1}{c} = \rho^2 \,. \end{cases}$$

$$(\partial_t + u\partial_x)\ln m = -2u_x \,,$$

or

$$m_t + 2mu_x + um_x = 0\,, \quad m = -u_{xx} + u + \frac{1}{c}\,.$$

Theorem

The DP equation

$$m_t + 3mu_x + um_x = 0\,, \quad m = \frac{1}{a} + u - u_{xx}$$

is derived from the bilinear equations

$$(D_x D_t + \frac{1}{a} D_x + a D_t) g \cdot f = 0 \qquad (11)$$

$$(\frac{1}{a} D_x + 1) g \cdot f = cF \qquad (12)$$

$$(\frac{1}{2} D_x D_t - 1) F \cdot F = -GG, \quad gf = cG \qquad (13)$$

through a hodograph transformation $x = ay + \ln \frac{g}{f}, \quad t = s$ *and dependent variable transformation* $u = \left(\ln \frac{g}{f}\right)_s$

Semi-discrete Camassa-Holm equation

The bilinear equations of the CH equation

$$(D_y D_s + \frac{1}{c} D_y + 2cD_s) g \cdot h = 0 \tag{14}$$

$$(\frac{1}{2c} D_y + 1) g \cdot h = ff \tag{15}$$

$$(\frac{1}{2} D_y D_s - 1) f \cdot f = -gh \tag{16}$$

Semi-discrete version of the CH equation

$$((1 + ac)D_s + a) g_{l+1} \cdot h_l - ((1 - ac)D_s + a) g_l \cdot h_{l+1} = 0 \tag{17}$$

$$(a + 1/c) g_{l+1} h_l + (a - 1/c) g_l h_{l+1} = 2a f_{l+1} f_l \tag{18}$$

$$(\frac{1}{a} D_s - 1) f_{l+1} \cdot f_l = -\frac{g_{l+1} h_l + g_l h_{l+1}}{2} \tag{19}$$

Continuous limits for $a \to 0$ are

$$(17)/a \to (14)$$

$$(18)/2a \to (15)$$

$$(19) \to (16)$$

The semi-discrete version of the bilinear equations

$$
\begin{aligned}
&((1+ac)D_s+a)g_{l+1}\cdot h_l-((1-ac)D_s+a)g_l\cdot h_{l+1}=0\\
&(a+1/c)g_{l+1}h_l+(a-1/c)g_lh_{l+1}=2af_{l+1}f_l\\
&(\frac{1}{a}D_s-1)f_{l+1}\cdot f_l=-\frac{g_{l+1}h_l+g_lh_{l+1}}{2}
\end{aligned}
$$

give the following semi-discrete CH equation

$$
\left\{
\begin{aligned}
&-2\left(\frac{w_{l+1}-w_l}{\delta_l}-\frac{w_l-w_{l-1}}{\delta_{l-1}}\right)+\delta_l\frac{w_{l+1}+w_l}{2}+\frac{\delta_l}{c}\frac{1-\dfrac{4a^2c^2}{\delta_l^2}}{1-a^2c^2}\\
&\qquad\qquad+\delta_{l-1}\frac{w_l+w_{l-1}}{2}+\frac{\delta_{l-1}}{c}\frac{1-\dfrac{4a^2c^2}{\delta_{l-1}^2}}{1-a^2c^2}=0\,,\\
&\frac{d\,\delta_l}{d\,t}=\left(1-\frac{\delta_l^2}{4}\right)(w_{l+1}-w_l)
\end{aligned}
\right.
$$

through transformations

$$
w_l=(\ln\frac{g_l}{h_l})_s\quad \delta_l=\frac{4af_{l+1}f_l}{(1/c+a)g_{l+1}h_l+(1/c-a)g_lh_{l+1}}
$$

Integrable self-adaptive mesh scheme for the Camassa-Holm equation

$$\begin{cases} \Delta^2 w_k = \dfrac{1}{\delta_k} M \left(\delta_k M w_k + \dfrac{1}{c\delta_k} \dfrac{\delta_k^2/c^2 - 4a^2}{1/c^2 - a^2} \right) , \\ \partial_t \delta_k = \left(1 - \dfrac{\delta_k^2}{4} \right) \delta_k \Delta w_k , \end{cases}$$

where $\Delta F_k = \frac{F_{k+1} - F_k}{\delta_k}$, $\quad M F_k = \frac{F_k + F_{k+1}}{2}$.

Time evolution of mesh: Modified forward Euler method and 4th order Runge-Kutta method

First equation: Solve the tridiagonal linear system by using the standard iteration method

Generalization of self-adaptive moving mesh method

- **Key idea:** the introduction of the hodograph transformation $(x, t) \to (y, s)$ based on the conservation law.
- Instead of the original PDEs, we consider the numerical solution of the coupled equation in terms of y and s.
- Note that the mesh density $\rho = \partial_x / \partial_y$ for the x-variable becomes nonuniform and time-dependent, which leads to a self-adaptive moving mesh scheme.
- The integrable discretization of the coupled dispersionless equation will lead to integrable self-adaptive moving mesh method.

General self-adaptive moving mesh methods for the short pulse equation

It was easily shown that the short pulse equation

$$\left(\sqrt{1+u_x^2}\right)_t - \left(\frac{1}{2}u^2\sqrt{1+u_x^2}\right)_x = 0$$

General self-adaptive moving mesh methods for the short pulse equation

It was easily shown that the short pulse equation

$$\left(\sqrt{1+u_x^2}\right)_t - \left(\frac{1}{2}u^2\sqrt{1+u_x^2}\right)_x = 0$$

Letting $\rho = \left(1+u_x^2\right)^{1/2}$, we can define a hodograph transformation

$$dy = \rho dx - \frac{1}{2}u^2\rho dt\,, \quad ds = dt\,,$$

the short pulse equation is transformed into the coupled dispersionless equation

$$\begin{cases} u_{ys} = \rho u, \\ \rho_s + \frac{1}{2}(u^2)_y = 0 \end{cases}$$

Then we can work on the structure-preserving schemes for the CD equation, which becomes self-adaptive moving mesh methods for the SP equation.

The generalized sine-Gordon equation and the self-adaptive moving method

The generalized sine-Gordon equation

$$u_{xt} = (1 + \sigma\partial_{xx}) \sin u\,, \quad u_{xt} - (\sigma(\cos u)u_x)_x = \sin u$$

or

$$u_{xt} - (\sigma \cos u)u_{xx} - (1 - \sigma u_x^2) \sin u = 0$$

can be written into a conservative form

$$\left(\sqrt{1 + u_x^2}\right)_t - \left(\sigma \cos u \sqrt{1 + u_x^2}\right)_x = 0$$

The generalized sine-Gordon equation and the self-adaptive moving method

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$$\left(\sqrt{1 + u_x^2}\right)_t - \left(\sigma \cos u \sqrt{1 + u_x^2}\right)_x = 0$$

Letting $\rho = \left(1 + u_x^2\right)^{1/2}$, we can define a hodograph transformation

$$dy = \rho dx + \sigma \cos u \rho dt \,, \quad ds = dt \,,$$

which means

$$\partial_y = \rho\partial_x \quad \partial_s = \partial_t + \cos u \partial_x$$

The generalized sine-Gordon equation and the self-adaptive moving mesh method

$$u_{xt} - (\sigma(\cos u)u_x)_x = \sin u\,, \quad \partial_x(\partial_t - \sigma \cos u \partial_x)u = \sin u$$

becomes

$$u_{ys} = \rho \sin u$$

The conservative form of the gsG equation

$$(\rho)_t - (\sigma \rho \cos u)_x = 0$$

becomes

$$\rho_s - (\sigma \cos u)_y = 0$$

In summary

$$\begin{cases} u_{ys} = \rho \sin u, \\ \rho_s - \sigma(\cos u)_y = 0 \end{cases}$$

A self-adaptive moving mesh method for the generalized sine-Gordon equation

Semi-discrete generalized sine-Gordon equation

$$\begin{cases} \frac{d}{ds}(u_{k+1}-u_k) = \frac{1}{2}(x_{k+1}-x_k)(\sin u_{k+1} + \sin u_k)\,, \\ \frac{d}{ds}(x_{k+1}-x_k) = \sigma(\cos u_{k+1} - \cos u_k)\,, \end{cases}$$

Self-adaptive moving mesh scheme

$$\begin{cases} p_k^{n+1} = p_k^n + \frac{1}{2}\delta_k^n(\sin u_{k+1}^n + \sin u_k^n)\Delta t\,, \\ \delta_k^{n+1} = \delta_k^n + \left(\cos u_{k+1}^{n+1} - \cos u_k^{n+1}\right)\sigma\Delta t\,, \end{cases}$$

where $p_k^n = u_{k+1}^n - u_k^n$, $\delta_k^n = x_{k+1}^n - x_k^n$.

General self-adaptive moving mesh methods for the CH equation

The conservative of the CH equation

$$m_t + 2mu_x + m_x u = 0, \quad m = \kappa + u - u_{xx}$$

$$\left(m^{\frac{1}{2}}\right)_t + \left(um^{\frac{1}{2}}\right)_x = 0$$

General self-adaptive moving mesh methods for the CH equation

The conservative of the CH equation

$$m_t + 2mu_x + m_x u = 0, \quad m = \kappa + u - u_{xx}$$

$$\left(m^{\frac{1}{2}}\right)_t + \left(um^{\frac{1}{2}}\right)_x = 0$$

Let $\rho = m^{-\frac{1}{2}}$, we can define a hodograph transformation

$$dy = \rho^{-1}dx - \rho^{-1}udt, \quad ds = dt\,,$$

The CH equation is transformed into the following coupled equation

$$\begin{cases} (\ln \rho)_{sy} \quad = \rho(\kappa + u) - \rho^{-1}, \\ \rho_s - u_y = 0 \end{cases}$$

General self-adaptive moving mesh methods for the DP equation

The conservative of the DP equation

$$m_t + 3mu_x + m_x u = 0, \quad m = \kappa + u - u_{xx}$$

$$\left(m^{\frac{1}{3}}\right)_t + \left(um^{\frac{1}{3}}\right)_x = 0$$

General self-adaptive moving mesh methods for the DP equation

The conservative of the DP equation

$$m_t + 3mu_x + m_x u = 0, \quad m = \kappa + u - u_{xx}$$

$$\left(m^{\frac{1}{3}}\right)_t + \left(um^{\frac{1}{3}}\right)_x = 0$$

Let $\rho = m^{-\frac{1}{3}}$, we can define a hodograph transformation

$$dy = \rho^{-1}dx - \rho^{-1}udt, \quad ds = dt\,,$$

The DP equation is transformed into the following coupled equation

$$\begin{cases} (\ln\rho)_{sy} \quad = \rho(\kappa + u) - \rho^{-2}, \\ \rho_s - u_y = 0 \end{cases}$$

General self-adaptive moving mesh methods for the b-family equation

The conservative of the b-family equation

$$m_t + bmu_x + m_x u = 0, \quad m = \kappa + u - u_{xx}$$

$$\left(m^{\frac{1}{b}}\right)_t + \left(um^{\frac{1}{b}}\right)_x = 0$$

General self-adaptive moving mesh methods for the b-family equation

The conservative of the b-family equation

$$m_t + bmu_x + m_x u = 0, \quad m = \kappa + u - u_{xx}$$

$$\left(m^{\frac{1}{b}}\right)_t + \left(um^{\frac{1}{b}}\right)_x = 0$$

Let $\rho = m^{-\frac{1}{b}}$, we can define a hodograph transformation

$$dy = \rho^{-1}dx - \rho^{-1}udt, \quad ds = dt\,,$$

The b-family equation is transformed into the following coupled equation

$$\begin{cases} (\ln\rho)_{sy} & = \rho(\kappa + u) - \rho^{-b+1}, \\ \rho_s - u_y = 0 \end{cases}$$

Conclusion and further topics

- A novel numerical method: integrable self-adaptive moving mesh method, is born from integrable discretizations of a class of soliton equations with hodograph transformation
- A self-adaptive moving mesh method is not necessarily to be integrable for integrable equations, which makes the task mcu easier
- A self-adaptive moving mesh method is not designed for integrable equations. It can be designed for other non-integrable equations based on hodograph transformation

Conclusion and further topics

- A novel numerical method: integrable self-adaptive moving mesh method, is born from integrable discretizations of a class of soliton equations with hodograph transformation
- A self-adaptive moving mesh method is not necessarily to be integrable for integrable equations, which makes the task mcu easier
- A self-adaptive moving mesh method is not designed for integrable equations. It can be designed for other non-integrable equations based on hodograph transformation
- Further topic 1: Detailed study and comparison of self-adaptive moving methods for several integrable equations mentioned here
- Further topic 2: Design ans study of self-adaptive moving mesh method for non-integrable nonlinear wave equations